THERMODYNAMICS OF NON-ABELIAN EXCLUSION STATISTICS

Wung-Hong Huang
Department of Physics
National Cheng Kung University
Tainan,70101,Taiwan

The thermodynamic potential of ideal gases described by the simplest non-abelian statistics is investigated. I show that the potential is the linear function of the element of the abelian-part statistics matrix. Thus, the factorizable property in the Haldane (abelian) fractional exclusion shown by the author [W. H. Huang, Phys. Rev. Lett. 81, 2392 (1998)] is now extended to the non-abelian case. The complete expansion of the thermodynamic potential is also given.

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E-mail: whhwung@mail.ncku.edu.tw

I. INTRODUCTION

It is know that the quasiparticle appearing in the strong correlation system in the low dimension may obeying the Haldane fractional exclusion statistics [1]. The famous examples are those in the fractional quantized Hall effect and spin 1/2 antiferromagnetic chain [2-4]. The thermodynamics in these system had been investigated by several authors [5-8].

Since the braid group in two space dimensions may be represented by the non-commute matrix, the non-abelian braid statistics could be shown in a real world [9-16]. These include the quasiparticle in the non-abelian fractional quantum Hall state as well as those in the conformal field theory [9-15]. The spinon in the Heisenberg chains could also show the non-abelian exchange statistics [16].

In a recent paper [15], Guruswamy and Schoutens had derived the occupation number distribution functions for the particle obeying the non-abelian statistics. They proposed the following equations for M types of particles and k types of pseudo-particles

$$(\frac{\Lambda_A - 1}{\Lambda_A}) \prod_B \Lambda_B^{\alpha_{AB}} \prod_i \Lambda_i^{G_{Ai}} = z_A, \qquad A = 1, ...M, \tag{1.1a}$$

$$(\frac{\Lambda_i - 1}{\Lambda_i}) \prod_A \Lambda_A^{G_{iA}} \prod_j \Lambda_j^{G_{ij}} = 1, \qquad i = 1, ...k-1, \tag{1.1b}$$

where α_{AB} is the abelian-part statistics matric, $G_{iA} = G_{Ai} = -\frac{1}{2}\delta_{i,1}$ and $G_{ij} = \frac{1}{2}(C_{k-1})_{ij}$, with C_{ij} the Cartan matrix of the associated group. Note that Λ_a is the single-level grand canonical partition function and $z_a \equiv e^{\beta(\mu_a - \epsilon)}$, with ϵ the energy level and μ_a the chemical potential of the particle of type a. When $G_{Ai} = 0$ Eq.(1.1a) describe the Haldane fractional exclusion statistics [5].

The Eqs.(1.1a) and (1.1b) are very similar to those in the abelian case [5]. However, the absence of the z_i $(i = 1...(k-1))$ in Eq.(1.1b) means that the k pseudo-particles do not be suppressed at high energy. This property is the characteristics of the non-abelian exclusion statistics [15].

In this letter I will analyze the case of (M,k)=(1,2). This is the simplest extension of the abelian to the non-abelian case. Despite the simplicity this case can describe the q-pfaffian non-abelian fractional quantum Hall state [12,13]. I will follow the method in my previous letter [7] to investigate the thermodynamic potential Q_α. A non-perturbative proof to is used to show that Q_α can be factorized in terms of these in the, so called, non-abelian boson $(\alpha = 0)$ and in the non-abelian fermion $(\alpha = 1)$, i.e.,

$$Q_\alpha(N) = (1-\alpha)Q_0(N) + \alpha Q_1(N). \tag{1.2}$$

Thus the factorizable property of the thermodynamical potential found in the Haldane (abelian) fractional exclusion statistics also appears in the non-abelian extension. As I also present the complete expansion of the thermodynamic potential.

II. FACTORIZATION AND VIRIAL EXPANSION

For simplest case (M,k)=(1,2) Eqs.(1.1a) and (1.1b) become

$$(\frac{\Lambda-1}{\Lambda})\Lambda^{\alpha}\Lambda_1^{-\frac{1}{2}} = z, \qquad (\frac{\Lambda_1-1}{\Lambda_1})\Lambda^{-\frac{1}{2}}\Lambda_1 = 1. \tag{2.1}$$

Eliminating Λ_1 from the above equation we have

$$(\frac{\Lambda-1}{\Lambda})\Lambda^{\alpha}(1+\Lambda^{\frac{1}{2}})^{-\frac{1}{2}} = z. \tag{2.2}$$

For later analysis the above equation is expressed as

$$\ln(\Lambda-1)+(\alpha-1)\ln\Lambda-\frac{1}{2}\ln(1+\Lambda^{\frac{1}{2}}) = \ln z = \beta(\mu-\epsilon). \tag{2.3}$$

Since the distribution function n is defined by [15]

$$n = \frac{d\ln\Lambda}{d\ln z}, \tag{2.4}$$

we can from Eq.(2.3) find that

$$n = \frac{1}{\frac{1}{4}\frac{1}{1+\Lambda^{\frac{1}{2}}}+\frac{1}{\Lambda-1}+(\alpha-\frac{1}{4})}. \tag{2.5}$$

Using this relation the differentiation of Eq.(2.3) with respect to α and ϵ will lead to two simple relations

$$\frac{d}{d\alpha}\ln\Lambda = n\ (\beta\frac{d\mu}{d\alpha}-\ln\Lambda), \tag{2.6}$$

$$\frac{d}{d\epsilon}\ln\Lambda = -n\beta, \tag{2.7}$$

respectively.

Using the above equation we see that

$$N \equiv \int_0^{\infty} d(\epsilon\beta)\frac{V}{\lambda^2}n = -\int_0^{\infty} d\epsilon\frac{V}{\lambda^2}\frac{d}{d\epsilon}\ln\Lambda = \frac{V}{\lambda^2}\ln\Lambda_0, \quad \Rightarrow \Lambda_0 = e^{\frac{N\lambda^2}{V}} \tag{2.8}$$

where N is the particle number and Λ_0 is the zero-energy grand canonical partition function. (Note that the system we considerd here has only one type of real particle, but two types of pseudo-particles. N denotes the total number of the real particles.) Now, substituting the above relation into Eq.(2.3) (letting $\epsilon = 0$) the exact form of the chemical potential can be found

$$\beta\mu = \alpha\frac{N\lambda^2}{V}+\ln(1-e^{-\frac{N\lambda^2}{V}})-\frac{1}{2}\ln(1+e^{\frac{1}{2}\frac{N\lambda^2}{V}}). \tag{2.9}$$

Thus the chemical potential μ is a linear function of the abelian-part statistics parameter α. Note that the first two term in the above equation are exactly those in the abelian exclusion statistics [5]. It is the last log term which becomes the new contribution in the non-abelian exclusion statistics.

Next, consider the differentiation of the thermodynamic potential Q_α with respect to α

$$\frac{dQ_\alpha}{d\alpha} = -kT\int_0^\infty d(\epsilon\beta)\frac{V}{\lambda^2}\frac{d}{d\alpha}\ln\Lambda = -kT\int_0^\infty d(\epsilon\beta)\frac{V}{\lambda^2}\; n\;(\beta\frac{d\mu}{d\alpha} - \ln\Lambda)$$

$$= -kT\int_0^\infty d(\epsilon\beta)\frac{V}{\lambda^2}\; n\;\frac{N\lambda^2}{V} - kT\int_0^\infty d\epsilon\frac{V}{\lambda^2}\ln\Lambda\frac{d}{d\epsilon}\ln\Lambda = -\frac{1}{2}NkT\frac{N\lambda^2}{V}, \tag{2.10}$$

in which the relations Eqs.(2.6)-(2.9) have been used. Thus the thermodynamical potential Q_α is a linear function of the abelian-part statistics parameter α. This means that the potential can be exprssed as $Q_\alpha = f + \alpha\, g$, in which f and g do not depend on the α. Then, if $\alpha = 1$ we find that $Q_1 = f + g$, and if $\alpha = 0$ we find that $Q_0 = f$. Thus $g = Q_1 - Q_0$ and $f = Q_0$ and we have the factorizable property in Eq.(1.2).

The remain work is to find the Virial expansion of the potential Q_α.

To do this we can first use the method of integration by part to replace the integration of ϵ by $\ln\Lambda$. Then, using the relations Eqs.(2.3), (2.8) and (2.9) we have

$$Q_\alpha = -kT\int_0^\infty d(\epsilon\beta)\frac{V}{\lambda^2}\ln\Lambda = kT\int_{\ln\Lambda_0}^0 d(\ln\Lambda)\frac{V}{\lambda^2}\epsilon\beta$$

$$= kT\int_{\ln\Lambda_0}^0 d(\ln\Lambda)\frac{V}{\lambda^2}\{\beta\mu - [\ln(\Lambda-1) + (\alpha-1)\ln\Lambda - \frac{1}{2}\ln(1+\Lambda^{\frac{1}{2}})]\}$$

$$= -kT(\ln\Lambda_0)\frac{V}{\lambda^2}\beta\mu - kT\int_{\ln\Lambda_0}^0 d(\ln\Lambda)\frac{V}{\lambda^2}[\ln(\Lambda-1) + (\alpha-1)\ln\Lambda - \frac{1}{2}\ln(1+\Lambda^{\frac{1}{2}})]$$

$$= -\frac{1}{2}\alpha NkT\frac{N\lambda^2}{V} - kT\int_0^{\frac{N\lambda^2}{V}} dx\frac{V}{\lambda^2}\frac{x}{e^x-1} + \frac{1}{4}kT\int_0^{\frac{N\lambda^2}{V}} dx\frac{V}{\lambda^2}\frac{xe^{x/2}}{e^{x/2}+1}, \tag{2.11}$$

in which the first integration can be expressed as the summation of the Bernoulli numbers $B_l(0)$, as that in the abelian case. The second integration is the new contribution from non-abelian statistics, which can be expressed as the summation of the Euler function $E_l(1)$. Using the mathematic definition

$$\frac{te^{xt}}{e^t-1} \equiv \sum_{l=0}^\infty B_l(x)\frac{t^l}{l!},$$

$$\frac{2e^{xt}}{e^t+1} \equiv \sum_{l=0}^\infty E_l(x)\frac{t^l}{l!}.$$

We thus find the final result

$$Q_\alpha = -\frac{1}{2}\alpha NkT\frac{N\lambda^2}{V} - NkT\;[\sum_{l=0}^\infty B_l(0)\frac{1}{(l+1)!}(\frac{N\lambda^2}{V})^l - \frac{1}{4}\sum_{l=0}^\infty E_l(1)\frac{l+1}{(l+2)!}(\frac{N\lambda^2}{2V})^{l+1}]$$

$$= -\frac{1}{2}\alpha NkT\frac{N\lambda^2}{V} - kTN[1 - \frac{5}{16}\frac{N\lambda^2}{V} + \frac{5}{288}(\frac{N\lambda^2}{V})^2 - \frac{17}{115200}(\frac{N\lambda^2}{V})^4$$

$$+\frac{13}{5419008}(\frac{N\lambda^2}{V})^6-\frac{257}{5573836800}(\frac{N\lambda^2}{V})^8+...], \tag{2.12}$$

which is the Virial expansion to the eighth order and is consistent with the Eq.(2.10).

It is seen that, as that in the abelian case [7], only the second Virial coefficient depends on the abelian-part Haldane statistics parameter. This means that only the zero-temperature quantities could depend on the Haldane statistics parameter.

III. CONCLUSION

In this paper I have investigated the thermodynamics of the simplest case, (M,k)=(1,2), of the non-abelian exclusion statistics. Despite its simplicity the case can be used to described the real world system. The q-pfaffian non-abelian fractional quantum Hall state is a special case with $\alpha=\frac{q+1}{4q}$ [12,13,15]. I have shown that the thermodynamic potential of the ideal gases obeying the simplest non-abelian exclusion statistics is a linear function of the (only) element of the abelian-part statistics matix. The factorizable property found in the abelian exclusion is thus now extended to the simplest non-abelian case. It is interesting to see that, in both of abelian and non-abelian case, only zero-temperature quantities could depend on the Haldane statistics parameter.

A little extension is that with arbitrary M (while remaining k=2). In this case $\Lambda_1=1+\prod_A\Lambda_A^{G_{1A}}$ (form Eq.(1.1b)). After substituting this into Eq.(1.1a) the pseudo-particle part is eliminated and thermodynamical potential could be studied with more analyses [8]. However, in the general system there may be more then one pseudo-particle, i.e., if $k>2$ in Eq.(1.1b). In this system the pseudo-particle part, Λ_i, could not be easily eliminated from the Eqs.(1.1a) and (1.1b). Thus the situation is more complex and the property of the thermodynamic potential is difficult to be analyzed. These problems are remained in my furthermore investigation.

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